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Hilbert–Enskog–Chapman Expansion in the Turbulent Kinetic Theory of Gases. I

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Giving up the assumption of molecular chaos, we get a hierarchy of equations, instead of the Boltzmann equation, governing the motion of a dilute gas. The Hilbert–Enskog–Chapman expansion methods for the hierarchy are studied in this paper and its continuations.

KEY WORDS: BBGGKY hierarchy; Hilbert uniqueness theorem; conservation equations; Enskog–Chapman expansion.

1. INTRODUCTION

In recent years Tsugé, Lewis, and others[(1–4)] began to study the theory of turbulence from the standpoint of the kinetic theory of gases. Grad[(5)] has pointed out that their works might shed a new light on the theory of turbulence. The method of averaging the Navier–Stokes equations, used in the classical theory of turbulence, has at least two disadvantages. First, logical confusion may be caused in averaging the Navier–Stokes equations, which are the results of a statistical averaging in the kinetic theory of gases themselves. Secondly, the problem of closing the infinite set of dynamical equations which couple together all the moments of the turbulent velocity field in the classical theory cannot be solved without some ad hoc assumptions, which are often more or less arbitrary. The new method used in the turbulent kinetic theory of gases is superior to that used in the classical theory, because it has succeeded in eliminating these two disadvantages. In deriving the macroscopic equations from the microscopic equations, Tsugé and Lewis used the generalized Grad's 13-moment method. Grad has pointed out that it is worthwhile to generalize the Hilbert–Enskog–

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Chapman expansion method, which is better founded, to the case of the turbulent kinetic theory of gases. This paper is a result of work along the line pointed out by Grad.

The paper will be divided into three parts, of which the first one is concerned with the following topics. With the help of the notion of truncated distributions we obtain a hierarchy of equations for the cumulant distributions. Since the hierarchy is slightly different from the well-known Bogoliubov–Born–Green–Kirkwood–Yvon (BBGKY) hierarchy and the notion of truncated distributions was first introduced by Grad in 1958, we call this hierarchy the BBGGKY hierarchy. We develop a generalized Hilbert expansion method for the BBGGKY hierarchy and get a Hilbert uniqueness theorem. The following two points are noteworthy. First, we (at least formally) make no assumption of molecular chaos at all, since the form of the BBGGKY hierarchy and the very definition of the expansion method have already absorbed, more or less, the essence of the assumption of molecular chaos. Secondly, the result of the expansion method for the Boltzmann equation is a special case of the results obtained in our general scheme. These two points apply to the generalized Enskog–Chapman expansion method as well. Having introduced the definition of the generalized Enskog–Chapman expansion method, we obtain the first-order approximate equations governing the motion of turbulent flows. There are altogether 30 partial differential equations governing 30 unknowns in our system of the first-order approximate macroscopic equations. On account of the symmetry of the correlation functions, the number of the equations can be reduced to 20.

In the second part of the paper we get a system of equations governing the evolution of the correlation functions as follows:

$$\begin{aligned}
&\frac{\partial R^{(1,1)}_{i,j}}{\partial t} + u_k \frac{\partial R^{(1,1)}_{i,j}}{\partial x_k} + \hat{u}_k \frac{\partial R^{(1,1)}_{i,j}}{\partial \hat{x}_k} + R^{(1,1)}_{k,j} \frac{\partial u_i}{\partial x_k} + R^{(1,1)}_{i,k} \frac{\partial \hat{u}_j}{\partial \hat{x}_k} \\
&\quad - \nu \Delta R^{(1,1)}_{i,j} - \nu \hat{\Delta} R^{(1,1)}_{i,j} + \frac{1}{3} \frac{\partial R^{(2,1)}_{j}}{\partial x_i} + \frac{1}{3} \frac{\partial R^{(1,2)}_{i}}{\partial \hat{x}_j} \\
&\quad - \kappa \frac{\partial}{\partial x_k} \left[\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) R^{(2,1)}_{j} \right] - \kappa \frac{\partial}{\partial \hat{x}_k} \left[\left(\frac{\partial \hat{u}_j}{\partial \hat{x}_k} + \frac{\partial \hat{u}_k}{\partial \hat{x}_j} \right) R^{(1,2)}_{i} \right] = 0
\end{aligned}$$

where κ is a new transport coefficient discovered by the generalized Enskog–Chapman expansion method. The above equations are similar in structure to the equations governing the evolution of the disturbances of the solutions of the Navier–Stokes equations, which have the following forms:

$$\frac{\partial v_i}{\partial t} + u_j \frac{\partial v_i}{\partial x_j} + v_j \frac{\partial u_i}{\partial x_j} - \nu \Delta u_i + \frac{\partial p}{\partial x_i} + u_j \frac{\partial u_i}{\partial x_j} = 0$$

The similarity between the structures of the two systems of equations offers us a possibility of constructing a theory for our system of equations similar to the classical stability theory for the Navier–Stokes equations and obtaining a new interpretation of the onset of turbulence.

In the third part of the paper we carry on with the Enskog–Chapman expansion further and discuss the notions of subcriticality and supercriticality.

2. HILBERT EXPANSION FOR THE BBGGKY HIERARCHY

The position of a particle will be denoted by x or x_i, $(i = 1,2,3)$, and its velocity by v or v_i $(i = 1,2,3)$. In a system of N particles we distinguish the positions of the particles by $x^1, \ldots, x^N$ and the same for the velocities $v^1, \ldots, v^N$. The $6N$-dimensional phase space with representative point $Z = (x^1, \ldots, x^n, v^1, \ldots, v^N)$ will be denoted by Γ, and the six-dimensional phase space with representative point $Z = (x, v)$ will be denoted by γ or γ_r if it refers specifically to the particle r with coordinates $z^r = (x^r, v^r)$; clearly $Z = (z^1, \ldots, z^N)$.

The motion of the system of n particles is governed by equations

$$\frac{dx^r}{dt} = v^r$$

$$m\frac{dv^r}{dt} = X^r$$

where m is the mass of a particle (molecule) and X^r is the force exerting on the particle r. In general we assume that there are no external forces. Hence

$$X^r = \sum_{s \neq r} X^{rs}(z^r, z^s)$$

where

$$X^{rs}(z^r, z^s) = X^{12}(z^r, z^s)$$

and

$$X^{12}(z^1, z^2) = \frac{r}{|r|}\phi'(|r|)$$

$$r = x^2 - x^1$$

We denote by $F_N(Z)$ a probability density defined on Γ; we have $F_N \geqslant 0$ and $\int F_N\, dZ = 1$. Generally we assume that F_N is symmetric with respect to the N groups of variables $z_1, \ldots, z_N$. The system of the equations governing the motion of the n particles defines a flow in the phase space Γ. The evolution of the probability density in time $F_N(Z, t)$ is determined once it is known initially. The equation governing $F_N(Z, t)$ is the so-called Liouville equation:

$$\frac{\partial F_N}{\partial t}+\sum_{r=1}^{N} v^r\cdot\frac{\partial F_N}{\partial x^r}+\frac{1}{m}\sum_{r=1}^{N} x^r\cdot\frac{\partial F_N}{\partial v^r}=0$$

where the dot "·" denotes the inner product of three-dimensional vectors.

The subdomain D_r^s of the space γ_r is defined by the inequality

$$|x^r-x^1|\geqslant\sigma,\qquad \ldots,\qquad |x^r-x^s|\geqslant\sigma\qquad (\text{all } V^r)$$

where σ is a small positive number, $s<r$, $x^1,\ldots,x^s$ are s given vectors in γ_r. Finally we set

$$R_r^s=D_r^s\times\cdots\times D_N^s$$

The truncated distribution function F_r^s is defined by the equality

$$F_r^s(z^1,\ldots,z^r)=\int_{R_{r+1}^s}F_n(z^1,\ldots,z^n)\,dz^{r+1}\cdots dz^n$$

where, $s\leqslant r$.

The gas is said to be dilute if the following relations hold:

$$F_r^r(z^1,\ldots,z^r)\approx F_r^{r-1}(z^1,\ldots,z^r)\qquad (r=1,2,\ldots)$$

For dilute gases F_r^r and F_r^{r-1} are considered to be identical and will be denoted by f_r/N^r:

$$f_r=N^rF_r^r=N^rF_r^{r-1}$$

Cumulant distribution functions $f, g, h, k,\ldots,$ are defined, respectively, by the following equalities:

$$f(z)=f_1(z)$$

$$g(z^1,z^2)=f_2(z^1,z^2)-f(z^1)f(z^2)$$

$$\begin{aligned}h(z^1,z^2,z^3)=\;&f_3(z^1,z^2,z^3)-f(z^1)f(z^2)f(z^3)-f(z^1)g(z^2,z^3)\\&-f(z^2)g(z^1,z^3)-f(z^3)g(z^1,z^2)\end{aligned}$$

$$\begin{aligned}k(z^1,z^2,z^3,z^4)=\;&f_4(z^1,z^2,z^3,z^4)-f(z^1)f(z^2)f(z^3)f(z^4)\\&-f(z^1)h(z^2,z^3,z^4)-f(z^2)h(z^1,z^3,z^4)\\&-f(z^3)h(z^1,z^2,z^4)-f(z^4)h(z^1,z^2,z^3)\\&-f(z^1)f(z^2)g(z^3,z^4)-f(z^1)f(z^3)g(z^2,z^4)\\&-f(z^1)f(z^4)g(z^2,z^3)-f(z^2)f(z^3)g(z^1,z^4)\\&-f(z^2)f(z^4)g(z^1,z^3)-f(z^3)f(z^4)g(z^1,z^2)\\&-g(z^1,z^2)g(z^3,z^4)-g(z^1,z^3)g(z^2,z^4)\\&-g(z^1,z^4)g(z^2,z^3)\end{aligned}$$

$$\cdots$$

Grad has formally derived the Boltzmann equation from the Liouville equation. In the process of deduction he has made four assumptions: (1) truncation, (2) binary collision, (3) molecular chaos, (4) slow variation of f_1. Among them the assumption of molecular chaos is the most controversial. Some people have tried to prove it, but no convincing result has been obtained. Recently Tsugé and others took another viewpoint: the assumption of molecular chaos is merely conditionally and approximately valid. Giving up the assumption of molecular chaos, with the help of the Grad method of deduction we can derive a hierarchy of equations instead of the Boltzmann equation. We call this hierarchy of equations the BBGGKY hierarchy. In terms of the cumulant distribution functions the first three equations in the hierarchy are

$$\left(\frac{\partial}{\partial t}+v_i\frac{\partial}{\partial x_i}\right)f(z,t)=J\left[f(z')f(\tilde{z}')-f(z)f(\tilde{z})\right]+J\left[g(z',\tilde{z}')-g(z,\tilde{z})\right] \tag{1}$$

$$\begin{aligned}\left(\frac{\partial}{\partial t}+v_i\frac{\partial}{\partial x_i}+\hat{v}_i\frac{\partial}{\partial \hat{x}_i}\right)g(z,\hat{z},t)=&\ \mathscr{J}[fg]_z+\mathscr{J}[fg]_{\hat{z}}\\&+J\left[h(z,\hat{z}',\tilde{z}')-h(z,\hat{z},\tilde{z})\right]_{\tilde{x}=\hat{x}}\\&+J\left[h(z',\hat{z},\tilde{z}')-h(z,\hat{z},\tilde{z})\right]_{\tilde{x}=x}\end{aligned} \tag{2}$$

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+v_i\frac{\partial}{\partial x_i}+\hat{v}_i\frac{\partial}{\partial \hat{x}_i}+\check{v}_i\frac{\partial}{\partial \check{x}_i}\right)h(z,\hat{z},\check{z},t)\\&\quad=\mathscr{J}[gg]_{z\hat{z}}+\mathscr{J}[gg]_{\hat{z}\check{z}}+\mathscr{J}[gg]_{\check{z}z}+\mathscr{J}[fh]_{z\hat{z}}+\mathscr{J}[fh]_{\hat{z}\check{z}}+\mathscr{J}[fh]_{\check{z}z}\\&\qquad+J\left[k(z,\hat{z},\check{z}',\tilde{z}')-k(z,\hat{z},\check{z},\tilde{z})\right]_{\tilde{x}=\check{x}}\\&\qquad+J\left[k(z',\hat{z},\check{z},\tilde{z}')-k(z,\hat{z},\check{z},\tilde{z})\right]_{\tilde{x}=x}\\&\qquad+J\left[k(z,\hat{z}',\check{z},\tilde{z}')-k(z,\hat{z},\check{z},\tilde{z})\right]_{\tilde{x}=\hat{x}}\end{aligned} \tag{3}$$

$$\cdots$$

where $z=(x,v)$, $\hat{z}=(\hat{x},\hat{v})$, . . . , are six-dimensional vectors; J denotes the collision integral operator:

$$J\left[g(z',\tilde{z}')-g(z,\tilde{z})\right]=\int\left[g(z',\tilde{z}')-g(z,\tilde{z})\right]B(\theta,U)\,d\theta\,d\epsilon\,d\tilde{v}$$

where $\tilde{x}=x$, $U=|\tilde{v}-v|$, and the meanings of θ and ϵ are standard.[6] In the case of Maxwellian molecules $B(\theta,U)$ is independent of U: $B(\theta,U)=B(\theta)$; the operator $\mathscr{J}$ is defined as follows:

$$\begin{aligned}\mathscr{J}[fg]_z=&\ J\left[f(\hat{z}')g(z,\tilde{z}')-f(\hat{z})g(z,\tilde{z})\right]_{\tilde{x}=\hat{x}}\\&+J\left[f(\tilde{z}')g(z,\hat{z}')-f(\tilde{z})g(z,\hat{z})\right]_{\tilde{x}=\hat{x}}\end{aligned}$$

interchanging the positions of z and $\hat{z}$ in the above equality, we shall get the expression of $[fg]_{\hat{z}}$:

$$\mathcal{J}[gg]_{z\hat{z}} = J[g(\check{z}',z)g(\tilde{z}',\hat{z}) - g(\check{z},z)g(\tilde{z},\hat{z})]_{\tilde{x}=\check{x}} + J[g(\tilde{z}',z)g(\check{z}',\hat{z}) - g(\tilde{z},z)g(\check{z},\hat{z})]_{\tilde{x}=\check{x}}$$

$\mathcal{J}[gg]_{\hat{z}\check{z}}$ and $\mathcal{J}[gg]_{\check{z}z}$ can be defined in a similar way:

$$\mathcal{J}[fh]_{z\hat{z}} = J[f(\check{z}')h(\tilde{z}',z,\hat{z}) - f(z)h(\tilde{z},z,\hat{z})]_{\tilde{x}=\check{x}} + J[f(\tilde{z}')h(\check{z}',z,\hat{z}) - f(\tilde{z})h(\check{z},z,\hat{z})]_{\tilde{x}=\check{x}}$$

$\mathcal{J}[fh]_{\hat{z}\check{z}}$ and $\mathcal{J}[fh]_{\check{z}z}$ can be defined similarly.

Now we are going to explain the Hilbert expansion method for the BBGGKY hierarchy (1), (2), (3), ... First, we multiply the left-hand sides of the equations (1), (2), (3), . . . , by a small parameter ϵ. Secondly, the unknown functions $f, g, h, k, \ldots,$ in the hierarchy are replaced, respectively, by the following power series in ϵ:

$$f = \sum_{n=N_0}^{\infty} \epsilon^n f^{(n)} \tag{4}$$

$$g = \sum_{n=N_1}^{\infty} \epsilon^n g^{(n)} \tag{5}$$

$$h = \sum_{n=N_2}^{\infty} \epsilon^n h^{(n)} \tag{6}$$

$$k = \sum_{n=N_3}^{\infty} \epsilon^n k^{(n)} \tag{7}$$

$$\cdots$$

where $N_0 < N_1 < N_2 < N_3 < \cdots$ is an increasing sequence of nonnegative integral numbers. The most interesting is the case $N_i = i$ $(i = 0, 1, 2, \ldots)$, which corresponds to the fluid motion with maximum correlations. For the case $N_0 = 0$ and N_1 being sufficiently large the Hilbert expansion method will give rise to the classical equations governing the fluid motion without correlations, e.g., the Euler equations, the Navier–Stokes equations, the Burnett equations, etc. In the sequel the calculation will be carried out only for the case $N_i = i$ $(i = 0, 1, 2, \ldots)$.

Comparing the coefficients of the powers in ϵ on both sides of the hierarchy, we obtain the following sequence of equations:

$$J[f^{(0)\prime}\tilde{f}^{(0)\prime} - f^{(0)}\tilde{f}^{(0)}] = 0 \tag{8}$$

$$\mathcal{J}[f^{(0)}g^{(1)}]_z + \mathcal{J}[f^{(0)}g^{(1)}]_{\hat{z}} = 0 \tag{9}$$

$$\mathcal{J}[f^{(0)}f^{(1)}] = \left(\frac{\partial}{\partial t} + v_i\frac{\partial}{\partial x_i}\right)f^{(0)} - J[g^{(1)}(z',\tilde{z}') - g^{(1)}(z,\tilde{z})] \tag{10}$$

$$\mathcal{J}[f^{(0)}h^{(2)}]_{z\hat{z}} + \mathcal{J}[f^{(0)}h^{(2)}]_{\hat{z}\tilde{z}} + \mathcal{J}[f^{(0)}h^{(2)}]_{\tilde{z}z}$$
$$= -\mathcal{J}[g^{(1)}g^{(1)}]_{z\hat{z}} - \mathcal{J}[g^{(1)}g^{(1)}]_{\hat{z}\tilde{z}} - \mathcal{J}[g^{(1)}g^{(1)}]_{\tilde{z}z} \tag{11}$$

$$\begin{aligned}\mathcal{J}[f^{(0)}g^{(2)}]_z + \mathcal{J}[f^{(0)}g^{(2)}]_{\hat{z}} = {} & \left(\frac{\partial}{\partial t} + v_i\frac{\partial}{\partial x_i} + \hat{v}_i\frac{\partial}{\partial \hat{x}_i}\right)g^{(1)} \\ & - J\left[h^{(2)}(z',\hat{z},\tilde{z}') - h^{(2)}(z,\hat{z},\tilde{z})\right]_{\tilde{x}=x} \\ & - J\left[h^{(2)}(z,\hat{z}',\tilde{z}') - h^{(2)}(z,\hat{z}',\tilde{z})\right]_{\tilde{x}=\hat{x}} \\ & - \mathcal{J}[f^{(1)}g^{(1)}]_z - \mathcal{J}[f^{(1)}g^{(1)}]_{\hat{z}}\end{aligned} \tag{12}$$

$$\begin{aligned}\mathcal{J}[f^{(0)}f^{(2)}] = {} & \left(\frac{\partial}{\partial t} + v_i\frac{\partial}{\partial x_i}\right)f^{(1)} - J[g^{(2)}(z',\tilde{z}') - g^{(2)}(z,\tilde{z})] \\ & - J[f^{(1)\prime}\tilde{f}^{(1)\prime} - f^{(1)}\tilde{f}^{(1)}]\end{aligned} \tag{13}$$

. . .

The Hilbert expansion method is the one solving equations (8), (9), (10), (11), (12), . . . , successively.

3. SUMMATIONAL INVARIANTS AND CONSERVATION EQUATIONS

It is well known that the solution of equation (8) is the local Maxwell distribution

$$f^{(0)} = n\left(\frac{m}{2\pi KT}\right)^{3/2}\exp(-m|v-u|^2/2KT)$$

where m is the mass of a molecule, K is the Boltzmann constant, and n, u, T are five functions of t and x (the vector u representing three functions).

The left-hand sides of equations (10), (13), . . . , are identical. The general solutions of their common homogeneous equation are of the form

$$f^{(0)}\sum_{0\leqslant i\leqslant 4} a_i\psi_i \tag{14}$$

where a_i are functions of t and x, and ψ_i are the (first-order) summational invariants:

$$\psi_0 = 1$$
$$\psi_i = v_i - u_i \qquad (1 \leqslant i \leqslant 3)$$
$$\psi_4 = |v-u|^2$$

The facts mentioned above are well known. The left-hand sides of equations (9), (12), . . . , are identical. The symmetric general solutions of their common homogeneous equation are of the form

$$f^{(0)}\hat{f}^{(0)} \sum_{0 \leqslant i, j \leqslant 4} a_{ij}\psi_i\hat{\psi}_j \tag{15}$$

where a_{ij} are functions of $t, x, \hat{x}$; obviously $a_{ij}(x, \hat{x}) = a_{ji}(\hat{x}, x)$, in particular, $a_{ij}(x, x) = a_{ji}(x, x)$. The equation (11) and the equation immediately after equation (13), etc., have identical left-hand sides. The symmetric general solutions of their common homogeneous equation are of the form

$$f^{(0)}\hat{f}^{(0)}\check{f}^{(0)} \sum_{0 \leqslant i, j, k \leqslant 4} a_{ijk}\psi_i\hat{\psi}_j\check{\psi}_k \tag{16}$$

where a_{ijk} are functions of $t, x, \hat{x}, \check{x}$; and a_{ijk} has a symmetry property similar to that of a_{ij}.

The proofs of these two facts (as well as their generalization to higher-order cases) are very simple. Now we are going to sketch the proof of the first fact as follows. Substituting the expression (15) for the function g in the equation

$$\mathcal{J}[f^{(0)}g]_z + \mathcal{J}[f^{(0)}g]_{\hat{z}} = 0$$

in virtue of the definition of J, the symmetry of J and the properties of the (first-order) summational invariants ψ_i, we can easily show that the above equality is satisfied. On the contrary, if g is a solution of the above equation, setting

$$g = f^{(0)}\hat{f}^{(0)}\phi$$

we have

$$f^{(0)}J\left[\hat{f}^{(0)}\tilde{f}^{(0)}(\phi(z, \tilde{z}') + \phi(z, \hat{z}') - \phi(z, \tilde{z}) - \phi(z, \hat{z}))\right]_{\tilde{x} = \hat{x}}$$
$$+ \hat{f}^{(0)}J\left[f^{(0)}\tilde{f}^{(0)}(\phi(\hat{z}, \tilde{z}') + \phi(\hat{z}, z') - \phi(\hat{z}, \tilde{z}) - \phi(\hat{z}, z))\right]_{\tilde{x} = x} = 0$$

In deriving the above equality, we have used the property of the Maxwellian distribution, $f^{(0)\prime}\tilde{f}^{(0)\prime} = f^{(0)}\tilde{f}^{(0)}$. Operating the integral operator $\int\phi(z, \hat{z}) \cdots dv\, d\tilde{v}$ on both sides of the above equality, on account of the symmetry property of the collision integral operator J, we have

$$\int f^{(0)}\hat{f}^{(0)}\tilde{f}^{(0)}\left\{\left[\phi(z, \tilde{z}') + \phi(z, \hat{z}') - \phi(z, \tilde{z}) - \phi(z, \hat{z})\right]^2\right.$$
$$\left. + \left[\phi(\hat{z}, \tilde{z}') + \phi(\hat{z}, z') - \phi(\hat{z}, \tilde{z}) - \phi(\hat{z}, z)\right]^2\right\}$$
$$\times B(\theta, U)\, d\theta\, d\epsilon\, dv\, d\hat{v}\, d\tilde{v} = 0$$

Hence,

$$\phi(z, \tilde{z}') + \phi(z, \hat{z}') - \phi(z, \tilde{z}) - \phi(z, \hat{z}) = 0$$
$$\phi(\hat{z}, \tilde{z}') + \phi(\hat{z}, z') - \phi(\hat{z}, \tilde{z}) - \phi(\hat{z}, z) = 0$$

Now it is an immediate consequence of these two equalities that ϕ is of the form

$$\phi = \sum_{0 \leqslant i, j \leqslant 4} a_{ij} \psi_i \hat{\psi}_j$$

We call the 25 functions $\psi_i \hat{\psi}_j$ ($0 \leqslant i, j \leqslant 4$) the second-order summational invariants, the 125 functions $\psi_i \hat{\psi}_j \check{\psi}_k$ ($0 \leqslant i, j, k \leqslant 4$) the third-order summational invariants.

Using the five first-order summational invariants, we get the following five conservation equations from equations (1) and (8):

$$\frac{Dn}{Dt} + n \frac{\partial u_i}{\partial x_i} = 0 \tag{17}$$

$$\frac{\partial P_{ij}}{\partial x_j} + mn \frac{Du_i}{Dt} = 0 \qquad (1 \leqslant i \leqslant 3) \tag{18}$$

$$\frac{DT}{Dt} + \frac{2}{3nK} \left(P_{ij} \frac{\partial u_i}{\partial x_j} + \frac{\partial q_i}{\partial x_j} \right) = 0 \tag{19}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i}$$

$$n = \int f \, dv$$

$$u_i = \frac{1}{n} \int v_i f \, dv$$

$$T = \frac{m}{3nK} \int |v - u|^2 f \, dv$$

$$P_{ij} = m \int (v_i - u_i)(v_j - u_j) f \, dv$$

$$q_i = \frac{m}{2} \int |v - u|^2 (v_i - u_i) f \, dv$$

Using the 25 second-order summational invariants, we can get 25 conservation equations in a similar way. In order to write down the conservation equations we introduce the following notations:

$$R^{(k, l)}_{i_1 \cdots i_k j_1 \cdots j_l} = \int g(z, \hat{z}, t) \prod_{s=1}^{k} (v_{i_s} - u_{i_s}) \prod_{r=1}^{l} (\hat{v}_{j_r} - \hat{u}_{j_r}) \, dv \, d\hat{v}$$

$$R^{(k+2, l)}_{i_1 \cdots i_k j_1 \cdots j_l} = R^{(k+2, l)}_{i_1 \cdots i_k m m j_1 \cdots j_l}$$

(the summation convention is applied on the right-hand side, $1 \leqslant m \leqslant 3$),

$$\frac{\mathfrak{D}}{\mathfrak{D}t} = \frac{\partial}{\partial t} + u_k \frac{\partial}{\partial x_k} + \hat{u}_k \frac{\partial}{\partial \hat{x}_k}$$

$$\frac{\hat{D}}{\hat{D}t} = \frac{\partial}{\partial t} + \hat{u}_k \frac{\partial}{\partial \hat{x}_k}$$

The 25 conservation equations are divided into nine classes. Because of the symmetry, we shall write down only six of them,

$$\frac{\mathscr{D} R^{(0,0)}}{\mathscr{D} t} + \left(\frac{\partial u_k}{\partial x_k} + \frac{\partial \hat{u}_k}{\partial \hat{x}_k} \right) R^{(0,0)} + \frac{\partial R^{(1,0)}_{k}}{\partial x_k} + \frac{\partial R^{(0,1)}_{k}}{\partial \hat{x}_k} = 0 \tag{20}$$

$$\frac{\mathscr{D} R^{(1,0)}_{i}}{\mathscr{D} t} + \left(\frac{\partial u_k}{\partial x_k} + \frac{\partial \hat{u}_k}{\partial \hat{x}_k} \right) R^{(1,0)}_{i} + \frac{D u_i}{D t} R^{(0,0)} + R^{(1,0)}_{k} \frac{\partial u_i}{\partial x_k}$$
$$+ \frac{\partial R^{(2,0)}_{ik}}{\partial x_k} + \frac{\partial R^{(1,1)}_{i,k}}{\partial \hat{x}_k} = 0 \qquad (1 \leqslant i \leqslant 3) \tag{21}$$

$$\frac{\mathscr{D} R^{(1,1)}_{i,j}}{\mathscr{D} t} + \left(\frac{\partial u_k}{\partial x_k} + \frac{\partial \hat{u}_k}{\partial \hat{x}_k} \right) R^{(1,1)}_{i,j} + \frac{D u_i}{D t} R^{(0,1)}_{j} + \frac{\hat{D} \hat{u}_j}{\hat{D} t} R^{(1,0)}_{i} + R^{(1,1)}_{k,j} \frac{\partial u_i}{\partial x_k}$$
$$+ R^{(1,1)}_{i,k} \frac{\partial \hat{u}_j}{\partial \hat{x}_k} + \frac{\partial R^{(2,1)}_{ik,j}}{\partial x_k} + \frac{\partial R^{(1,2)}_{i,jk}}{\partial \hat{x}_k} = 0 \qquad (1 \leqslant i, j \leqslant 3) \tag{22}$$

$$\frac{\mathscr{D} R^{(2,0)}}{\mathscr{D} t} + \left(\frac{\partial u_k}{\partial x_k} + \frac{\partial \hat{u}_k}{\partial \hat{x}_k} \right) R^{(2,0)} + 2 \frac{D u_i}{D t} R^{(1,0)}_{i} + 2 R^{(2,0)}_{ik} \frac{\partial u_i}{\partial x_k} + \frac{\partial R^{(3,0)}_{k}}{\partial x_k}$$
$$+ \frac{\partial R^{(2,1)}_{k}}{\partial \hat{x}_k} = 0 \tag{23}$$

$$\frac{\mathscr{D} R^{(2,1)}_{j}}{\mathscr{D} t} + \left(\frac{\partial u_k}{\partial x_k} + \frac{\partial \hat{u}_k}{\partial \hat{x}_k} \right) R^{(2,1)}_{j} + 2 \frac{D u_i}{D t} R^{(1,1)}_{i,j} + \frac{\hat{D} \hat{u}_j}{\hat{D} t} R^{(2,0)} + 2 R^{(2,1)}_{ik,j} \frac{\partial u_i}{\partial x_k}$$
$$+ R^{(2,1)}_{k} \frac{\partial \hat{u}_j}{\partial \hat{x}_k} + \frac{\partial R^{(3,1)}_{k,j}}{\partial x_k} + \frac{\partial R^{(2,2)}_{jk}}{\partial \hat{x}_k} = 0 \qquad (1 \leqslant j \leqslant 3) \tag{24}$$

$$\frac{\mathscr{D} R^{(2,2)}}{\mathscr{D} t} + \left(\frac{\partial u_k}{\partial x_k} + \frac{\partial \hat{u}_k}{\partial \hat{x}_k} \right) R^{(2,2)} + 2 \frac{D u_i}{D t} R^{(1,2)}_{i} + 2 \frac{\hat{D} \hat{u}_j}{\hat{D} t} R^{(2,1)}_{j}$$
$$+ 2 R^{(2,2)}_{ik} \frac{\partial u_i}{\partial x_k} + 2 R^{(2,2)}_{jk} \frac{\partial \hat{u}_j}{\partial \hat{x}_k} + \frac{\partial R^{(3,2)}_{k}}{\partial x_k} + \frac{\partial R^{(2,3)}_{k}}{\partial \hat{x}_k} = 0 \tag{25}$$

Using the 125 third-order summational invariants we can derive 125 conservation equations from equation (3). These conservation equations will be published in the continuation (II) of this paper.

4. A GENERALIZATION OF THE HILBERT UNIQUENESS THEOREM

The solution of equation (8) is the local Maxwell distribution, in which the five functions n, u, T are undetermined. In order to solve equations

(9)–(13), etc., we have to make the following substitutions; e.g., for the unknown function $g^{(2)}$ the substitution is

$$g^{(12)} = f^{(0)}\hat{f}^{(0)}\phi$$

Having made the substitution, we have the left-hand side of equation (12) a non-positive-definite symmetric operator on ϕ. Zero is one of its eigenvalues, of which the corresponding eigensubspace is the 25-dimensional subspace spanned over 25 second-order summational invariants. Using the results obtained in the articles of Refs. 9 and 10, we can easily show that the non-positive-definite symmetric operator mentioned above is negative definite on the orthogonal complement of the 25-dimensional eigensubspace. Using the spectral theorem for symmetric operators,[11] we can show that a necessary and sufficient condition for solvability of equation (12) is the orthogonality of the right-hand side of equation (12) to the 25 second-order summational invariants. It is well known that the 24 second-order summational invariants are orthogonal to the terms J and $\mathcal{J}$ on the right-hand side of equation (12). Therefore a necessary and sufficient condition for solvability of equation (12) is

$$\int \psi_i \hat{\psi}_j \left(\frac{\partial}{\partial t} + v_k \frac{\partial}{\partial x_k} + \hat{v}_k \frac{\partial}{\partial \hat{x}_k} \right) g^{(1)}\, dv\, d\hat{v} = 0 \tag{26}$$

The above argument still holds good for the other equations in (8)–(13), etc. In case the condition of solvability is satisfied, the solution is not unique and the general solution of an equation is the sum of a special solution of the equation and the general solution of the corresponding homogeneous equation. So we have

$$f^{(0)} = n\left(\frac{m}{2\pi KT} \right)^{3/2} \exp(-m|v-u|^2/2KT)$$

$$f^{(n)} = \bar{f}^{(n)} + f^{(0)} \sum_{0 \leqslant i \leqslant 4} \alpha_i^n \psi_i \qquad (n \geqslant 1)$$

$$g^{(n)} = \bar{g}^{(n)} + f^{(0)}\hat{f}^{(0)} \sum_{0 \leqslant i,j \leqslant 4} \beta_{ij}^n \psi_i \hat{\psi}_j \qquad (n \geqslant 1)$$

$$h^{(n)} = \bar{h}^{(n)} + f^{(0)}\hat{f}^{(0)}\check{f}^{(0)} \sum_{0 \leqslant i,j,k \leqslant 4} \gamma_{ijk}^n \psi_i \hat{\psi}_j \check{\psi}_k \qquad (n \geqslant 2)$$

$$\cdots$$

where $\bar{f}^{(n)}, \bar{g}^{(n)}, \bar{h}^{(n)}, \ldots,$ are special solutions; and $n, u, T, \alpha_i^n, \beta_{ij}^n, \gamma_{ijk}^n, \ldots,$ are functions in time variable and position variables. For the sake of definiteness we always require the special solutions $\bar{f}^{(n)}, \bar{g}^{(n)},$

$\bar{h}^{(n)}, \ldots,$ to satisfy, respectively, the following conditions:

$$\int \bar{f}^{(n)}\psi_i \, dv = 0 \qquad (n \geqslant 1, 0 \leqslant i \leqslant 4)$$

$$\int \bar{g}^{(n)}\psi_i\hat{\psi}_j \, dv \, d\hat{v} = 0 \qquad (n \geqslant 1, 0 \leqslant i, j \leqslant 4)$$

$$\int \bar{h}^{(n)}\psi_i\hat{\psi}_j\check{\psi}_k \, dv \, d\hat{v} \, d\check{v} = 0 \qquad (n \geqslant 2, 0 \leqslant i, j, k \leqslant 4)$$

$$\cdots$$

In order that the equations (8)–(13), etc. be solvable, we shall have a sequence of equations like equation (26). They are a system of partial differential equations in unknowns $n, u, T, \alpha_i^n, \beta_{ij}^n, \gamma_{ijk}^n, \ldots,$ etc. The partial derivatives of the unknowns with respect to t in these equations are of first order. They can be solved from the equations. Using the uniqueness of the solution of the Cauchy problem of the partial differential equations we have the following theorem.

Hilbert Uniqueness Theorem. After the left-hand sides of equations (1), (2), (3), ... are multiplied by ϵ, the solutions of the BBGGKY hierarchy of the form

$$f = \sum_{n=0}^{\infty} \epsilon^n f^{(n)}$$

$$g = \sum_{n=1}^{\infty} \epsilon^n g^{(n)}$$

$$h = \sum_{n=2}^{\infty} \epsilon^n h^{(n)}$$

$$\cdots$$

are uniquely determined by the values of the integrals of the products of these solutions and their respective summational invariants over the velocity space at $t = 0$.

5. ENSKOG–CHAPMAN EXPANSION FOR THE BBGGKY HIERARCHY

It is very difficult to use the Hilbert expansion method to obtain some exact results, since we shall encounter a formidable obstacle in the way of carrying out the calculation, i.e., solving several systems of nonlinear partial differential equations. Enskog and Chapman have cleverly made a detour to bypass the obstacle. It is an immediate consequence of the skillful design of their expansion method that the solvability conditions for the integral equations are automatically satisfied. Hence we need not solve systems of nonlinear partial differential equations. The steps of the Enskog–Chapman expansion method for the BBGGKY hierarchy are as follows: (i) multiply the left-hand sides of equations (1), (2), (3), . . . , by ϵ; (ii) replace $\partial/\partial t$ by $\sum_{h=0}^{\infty}\epsilon^n \partial_n/\partial t$ (the meanings of the operators $\partial_n/\partial t$ will be explained later on); (iii) substitute the following power series

$$f = \sum_{n=N_0}^{\infty} \epsilon^n f^{(n)}$$

$$g = \sum_{n=N_1}^{\infty} \epsilon^n g^{(n)}$$

$$h = \sum_{n=N_2}^{\infty} \epsilon^n h^{(n)}$$

$$\cdots$$

for the unknowns $f, g, h, \ldots$ in the BBGGKY hierarchy, where $N_0 < N_1 < N_2 < \cdots$ is an increasing sequence of nonnegative integers; (iv) comparing the coefficients of the powers of ϵ on both sides of the equations we have just obtained, we derive a sequence of equations like (8)–(13), etc.; (v) solve the equations in the sequence with additional integral conditions:

$$\int f^{(N_0)}\psi_i \, dv = \int f\psi_i \, dv \qquad (0 \leqslant i \leqslant 4)$$

$$\int f^{(n)}\psi_i \, dv = 0 \qquad (0 \leqslant i \leqslant 4, n > N_0)$$

$$\int g^{(N_1)}\psi_i\hat{\psi}_j \, dv \, d\hat{v} = \int g\psi_i\hat{\psi}_j \, dv \, d\hat{v} \qquad (0 \leqslant i, j \leqslant 4)$$

$$\int g^{(n)}\psi_i\hat{\psi}_j \, dv \, d\hat{v} = 0 \qquad (0 \leqslant i, j \leqslant 4, n > N_1)$$

$$\int h^{(N_2)}\psi_i\hat{\psi}_j\check{\psi}_k \, dv \, d\hat{v} \, d\check{v} = \int h\psi_i\hat{\psi}_j\check{\psi}_k \, dv \, d\hat{v} \, d\check{v} \qquad (0 \leqslant i, j, k \leqslant 4)$$

$$\int h^{(n)}\psi_i\hat{\psi}_j\check{\psi}_k \, dv \, d\hat{v} \, d\check{v} = 0 \qquad (0 \leqslant i, j, k \leqslant 4, n > N_2)$$

$$\cdots$$

Now we are going to explain the meanings of the operators $\partial_n/\partial t$ as follows: The results of the operators $\partial_n/\partial t$ operating on the 25 R's will be formally given with the help of the conservation equations in the same way as $\partial_n n/\partial t$, $\partial_n u_i/\partial t$, and $\partial_n T/\partial t$ were.[8] For example,

$$\frac{\partial_0 R^{(2,2)}}{\partial t} = -u_k \frac{\partial R^{(2,2)}}{\partial x_k} - \hat{u}_k \frac{\partial R^{(2,2)}}{\partial \hat{x}_k} - \left(\frac{\partial u_k}{\partial x_k} + \frac{\partial \hat{u}_k}{\partial \hat{x}_k} \right) R^{(2,2)}$$
$$- 2 \frac{D_0 u_i}{Dt} R_i^{(1,2)} - 2 \frac{\hat{D}_0 \hat{u}_j}{Dt} R_j^{(2,1)} - 2R_{ik}^{(2,2)(0)} \frac{\partial u_i}{\partial x_k} - 2R_{jk}^{(2,2)(0)} \frac{\partial \hat{u}_j}{\partial \hat{x}_k}$$
$$- \frac{\partial R_k^{(3,2)(0)}}{\partial x_k} - \frac{\partial R_k^{(2,3)(0)}}{\partial \hat{x}_k}$$

$$\frac{\partial_n R^{(2,2)}}{\partial t} = -2R_{ik}^{(2,2)(n)} \frac{\partial u_i}{\partial x_k} - 2R_{jk}^{(2,2)(n)} \frac{\partial \hat{u}_j}{\partial \hat{x}_k} - \frac{\partial R_k^{(3,2)(n)}}{\partial x_k}$$
$$- \frac{\partial R_k^{(2,3)(n)}}{\partial \hat{x}_k} - 2 \frac{\partial_n u_i}{\partial t} R_i^{(1,2)} - 2 \frac{\hat{\partial}_n \hat{u}_j}{\hat{\partial} t} R_j^{(2,1)} \qquad (n \geqslant 1)$$

where

$$R_k^{(3,2)(n)} = \sum_{1 \leqslant i, j \leqslant 3} \int g^{(n+1)} (v_i - u_i)^2 (v_k - u_k)(\hat{v}_j - \hat{u}_j)^2 \, dv \, d\hat{v}$$

$R_{ik}^{(2,2)(n)}$, $R_{jk}^{(2,2)(n)}$, $R_k^{(2,3)(n)}$ can be defined similarly. The results of the operator $\partial_n/\partial t$ operating on the derivatives of the R's with respect to the space variables x_i can be defined by formally exchanging $\partial_n/\partial t$ and $\partial/\partial x_i$. If f is a function of n, u, T, R's and their derivatives, $\partial_n f/\partial t$ is formally defined by the chain rule.

The Enskog–Chapman expansion method gives us the following sequence of solvable integral equations:

$$J[f^{(0)'} \tilde{f}^{(0)'} - f^{(0)} \tilde{f}^{(0)}] = 0 \tag{27}$$

$$\mathcal{J}[f^{(0)} g^{(1)}]_z + \mathcal{J}[f^{(0)} g^{(1)}]_{\hat{z}} = 0 \tag{28}$$

$$\mathcal{J}[f^{(0)} f^{(1)}] = \left(\frac{\partial_0}{\partial t} + v_i \frac{\partial}{\partial x_i} \right) f^{(0)} - J[g^{(1)}(z', \tilde{z}') - g^{(1)}(z, \tilde{z})] \tag{29}$$

$$\mathcal{J}[f^{(0)} h^{(2)}]_{z\hat{z}} + \mathcal{J}[f^{(0)} h^{(2)}]_{\hat{z}\tilde{z}} + \mathcal{J}[f^{(0)} h^{(2)}]_{\tilde{z}z}$$
$$= -(\mathcal{J}[g^{(1)} g^{(1)}]_{z\hat{z}} + \mathcal{J}[g^{(1)} g^{(1)}]_{\hat{z}\tilde{z}} + \mathcal{J}[g^{(1)} g^{(1)}]_{\tilde{z}z}) \tag{30}$$

$$\mathcal{J}[f^{(0)} g^{(2)}]_z + \mathcal{J}[f^{(0)} g^{(2)}]_{\hat{z}} = \left(\frac{\partial_0}{\partial t} + v_i \frac{\partial}{\partial x_i} + \hat{v}_i \frac{\partial}{\partial \hat{x}_i} \right) g^{(1)}$$
$$- J[h^{(2)}(z' \hat{z}, \tilde{z}') - h^{(2)}(z, \hat{z}, \tilde{z})]_{\tilde{x}=x}$$
$$- J[h^{(2)}(z, \hat{z}', \tilde{z}') - h^{(2)}(z, \hat{z}, \tilde{z})]_{\tilde{x}=\hat{x}}$$
$$- \mathcal{J}[f^{(1)} g^{(1)}]_z - \mathcal{J}[f^{(1)} g^{(1)}]_{\hat{z}} \tag{31}$$

$$\mathcal{J}[f^{(0)}f^{(2)}] = \left(\frac{\partial_0}{\partial t} + v_i\frac{\partial}{\partial x_i}\right)f^{(1)} + \frac{\partial_1}{\partial t}f^{(0)}$$
$$-J[g^{(2)}(z',\tilde{z}') - g^{(2)}(z,\tilde{z})] - J[f^{(1)'}\tilde{f}^{(1)'} - f^{(1)}\tilde{f}^{(1)}] \quad (32)$$
$$\cdots$$

The solutions of equations (27) and (28) are well known. The solution of equation (29) and its corresponding macroscopic equations will be given in the rest of the paper. The solutions of equations (30) and (31) will be published in the subsequent article (II), and the solution of equation (32) in the subsequent article (III).

6. THE SOLUTION OF EQUATION (29)

The solution of equation (27) is the local Maxwell distribution

$$f^{(0)} = n\left(\frac{m}{2\pi KT}\right)^{3/2}\exp(-m|v-u|^2/2KT) \quad (33)$$

where n, u, and T are exactly the same functions as in f.

A solution $g^{(1)}$ of equation (28) is a product of $f^{(0)}\hat{f}^{(0)}$ and a linear combination of second-order summational invariants:

$$g^{(0)} = f^{(0)}\hat{f}^{(0)}\sum_{0\leqslant i,j\leqslant 4} a_{ij}\psi_i\hat{\psi}_j \quad (34)$$

where $f^{(0)}$ is the local Maxwell distribution (33), and a_{ij} are functions in $t, x, \hat{x}$. Because of the symmetry of $g^{(1)}$ we have $a_{ij}(t,x,\hat{x}) = a_{ji}(t,\hat{x},x)$. On account of the additional integral condition of the solution of the equation we have the following relations between the a_{ij}'s and the R's:

$$a_{ij} = \frac{m^2}{n\hat{n}K^2T\hat{T}}R^{(1,1)}_{i,j} \qquad (1\leqslant i,j\leqslant 3) \quad (35)$$

$$a_{i0} = \frac{m^2}{2n\hat{n}K^2T\hat{T}}\left(\frac{5K\hat{T}}{m}R^{(1,0)}_{i} - R^{(1,2)}_{i}\right) \qquad (1\leqslant i\leqslant 3) \quad (36)$$

$$a_{i4} = \frac{m^3}{6n\hat{n}K^3T\hat{T}^2}\left(R^{(1,2)}_{i} - \frac{3K\hat{T}}{m}R^{(1,0)}_{i}\right) \qquad (1\leqslant i< 3) \quad (37)$$

$$a_{00} = \frac{m^2}{4n\hat{n}K^2T\hat{T}}\left(\frac{25K^2T\hat{T}}{m^2}R^{(0,0)} - \frac{5K\hat{T}}{m}R^{(2,0)} - \frac{5KT}{m}R^{(0,2)} + R^{(2,2)}\right) \quad (38)$$

$$a_{04} = \frac{m^3}{12n\hat{n}K^3T\hat{T}^2}\left(-\frac{15K^2T\hat{T}}{m^2}R^{(0,0)} + \frac{3K\hat{T}}{m}R^{(2,0)} + \frac{5KT}{m}R^{(0,2)} - R^{(2,2)}\right) \quad (39)$$

$$a_{44} = \frac{m^4}{36 n\hat{n} K^4 T^2 \hat{T}^2} \left(\frac{9K^2 T\hat{T}}{m^2} R^{(0,0)} - \frac{3K\hat{T}}{m} R^{(2,0)} - \frac{3KT}{m} R^{(0,2)} + R^{(2,2)} \right) \tag{40}$$

the expressions of a_{0i}, a_{4i}, and a_{40} can be given in a symmetric way.

In order to solve equation (29) we must calculate the right-hand side of the equation. The first term on the right-hand side of the equation is well known[(7-8)]:

$$\left(\frac{\partial_0}{\partial t} + v_i \frac{\partial}{\partial x_i} \right) f^{(0)} = f^{(0)} \left\{ \frac{1}{2T} \frac{\partial T}{\partial x_i} (v_i - u_i) \left(\frac{m|v-u|^2}{KT} - 5 \right) + \frac{m}{KT} \frac{\partial u_i}{\partial x_j} \left[(v_i - u_i)(v_j - u_j) - \frac{1}{3} |v-u|^2 \delta_{ij} \right] \right\} \tag{41}$$

$$\begin{aligned} J\left[g^{(1)}(z', \tilde{z}') - g^{(1)}(z, \tilde{z}) \right] &= \sum_{0 \leqslant i, j \leqslant 4} a_{ij} J\left[f^{(0)} \tilde{f}^{(0)} (\psi_i' \tilde{\psi}_j' - \psi_i \tilde{\psi}_j) \right] \\ &= \sum_{1 \leqslant i < j \leqslant 4} a_{ij} J\left[f^{(0)} \tilde{f}^{(0)} (\psi_i' \tilde{\psi}_j' + \tilde{\psi}_i' \psi_j' - \psi_i \tilde{\psi}_j - \tilde{\psi}_i \psi_j) \right] \\ &\quad + \sum_{1 \leqslant i \leqslant 4} a_{ii} J\left[f^{(0)} \tilde{f}^{(0)} (\psi_i' \tilde{\psi}_i' - \psi_i \tilde{\psi}_i) \right] \end{aligned} \tag{42}$$

where $a_{ij} = a_{ij}(x, x)$. In the derivation of the above equality we have used two simple facts: $x = \tilde{x}$ in the operator J, hence $a_{ij} = a_{ji}$; and since $\psi_0 = \tilde{\psi}_0 = \psi_0' = \tilde{\psi}_0' = 1$, we have

$$\psi_0' \tilde{\psi}_i' + \tilde{\psi}_0 \psi_i' - \psi_0 \tilde{\psi}_i - \tilde{\psi}_0 \psi_i = 0$$

$$\psi_0' \tilde{\psi}_0' - \psi_0 \tilde{\psi}_0 = 0$$

For the sake of convenience we introduce the following notations:

$$w = v - u, \qquad \tilde{w} = \tilde{v} - u, \qquad w' = v' - u, \qquad \tilde{w}' = \tilde{v}' - u$$

where the function u is exactly the same function as in $f^{(0)}$. From now on we assume that the molecules are Maxwellian; therefore, the collision integral operator J is of the form

$$J[\cdots] = \int \cdots B(\theta)\, d\theta\, d\epsilon\, d\tilde{v}$$

where the meanings of θ and ϵ are standard.[(6)] The following notations are well known:

$$U = \tilde{w} - w, \qquad \alpha A = w' - w, \qquad |\alpha| = 1, \qquad A = \alpha \cdot U$$

In a suitable coordinate system the coordinates of U and α will be $(|U|, 0, 0)$ and $(\cos\theta, \sin\theta\cos\epsilon, \sin\theta\sin\epsilon)$, respectively, where θ and ϵ are exactly the same variables as in the collision integral. The relations between the molecular velocities before and after the collision are of the form

$$w' = w + \alpha(\alpha \cdot U)$$
$$\tilde{w}' = \tilde{w} - \alpha(\alpha \cdot U)$$

Now we are going to calculate the right-hand side of the equality (42). The terms on the right-hand side can be calculated one by one as follows:

$$\begin{aligned} w'\tilde{w}' + \tilde{w}'w' - w\tilde{w} - \tilde{w}w &= \left[w + \alpha(\alpha \cdot U)\right]\left[\tilde{w} - \alpha(\alpha \cdot U)\right] \\ &\quad + \left[\tilde{w} - \alpha(\alpha \cdot U)\right]\left[w + \alpha(\alpha \cdot U)\right] - w\tilde{w} - \tilde{w}w \\ &= (\alpha \cdot U)(\alpha\tilde{w} + \tilde{w}\alpha - \alpha w - w\alpha) - 2(\alpha \cdot U)^2\alpha\alpha \end{aligned}$$

where $w\tilde{w}$ denotes the dyadic product of w and $\tilde{w}$.

Using the following easily verified equalities

$$\int \alpha\, d\epsilon = \frac{2\pi\cos\theta}{|U|}\, U \tag{43}$$

$$\int \alpha\alpha\, d\epsilon = \pi\sin^2\theta\delta + 2\pi\left(1 - \frac{3}{2}\sin^2\theta\right)\frac{UU}{|U|^2} \tag{44}$$

we have

$$\begin{aligned} &\int \left[w'\tilde{w}' + \tilde{w}'w' - w\tilde{w} - \tilde{w}w\right] d\epsilon \\ &\quad = 2\pi\cos^2\theta(U\tilde{w} + \tilde{w}U - Uw - wU) \\ &\quad\quad - 2|U|^2\cos^2\theta\left[\pi\sin^2\theta \cdot \delta + 2\pi\left(1 - \frac{3}{2}\sin^2\theta\right)\frac{UU}{|U|^2}\right] \end{aligned}$$

where δ is the second-order unit tensor and the integral $\int \cdots d\epsilon = \int_0^{2\pi} \cdots d\epsilon$. The right-hand side of the above equality consists of two terms. The first term is a multiple of $\cos^2\theta\sin^2\theta$ and the second a multiple of $\cos^2\theta$. A simple calculation will show us that the last term vanishes. Rather than make the direct calculation, we would show it in the following way.

Lemma. If f and g are two (sufficiently good) functions in three variables and

$$K(\theta, w, w) = \int \left[f(w')g(\tilde{w}') + f(\tilde{w}')g(w') - f(w)g(\tilde{w}) - f(\tilde{w})g(w)\right] d\epsilon$$

we have

$$K(\theta + \tfrac{1}{2}\pi, w, \tilde{w}) = K(\theta, w, \tilde{w})$$

Proof. Denote the vectors α, w', $\tilde{w}'$ for the parameter $\theta + \frac{1}{2}\pi$ by $\bar{\alpha}$, $\bar{w}'$, $\bar{\tilde{w}}'$, respectively. Evidently, $\bar{\alpha} = (-\sin\theta, \cos\theta\cos\epsilon, \cos\theta\sin\epsilon)$. A simple calculation will show us that $\bar{w}' = \tilde{w}'$, $\bar{\tilde{w}}' = w'$, and the lemma follows.

Because $\cos^2(\theta + \frac{1}{2}\pi)\sin^2(\theta + \frac{1}{2}\pi) = \cos^2\theta\sin^2\theta$ and $\cos^2(\theta + \frac{1}{2}\pi) \neq \cos^2\theta$ (unless $\theta = \frac{1}{4}\pi + \frac{1}{2}k\pi$), we have

$$\int (w'\tilde{w}' + \tilde{w}'w' - w\tilde{w} - \tilde{w}w)\, d\epsilon$$
$$= 2\pi\cos^2\sin^2\theta(3UU - |U|^2\delta)$$
$$= 2\pi\cos^2\theta\sin^2\theta\left[3(\tilde{w}\tilde{w} + ww - \tilde{w}w - w\tilde{w}) - (|w|^2 + |\tilde{w}|^2 - 2w\cdot\tilde{w})\delta\right]$$

Hence,

$$\int f^{(0)}\tilde{f}^{(0)}\left[w'\tilde{w}' + \tilde{w}'w' - w\tilde{w} - \tilde{w}w\right]B(\theta)\,d\theta\,d\epsilon\,d\tilde{v}$$
$$= 2B_1 nf^{(0)}(3ww - |w|^2\delta) \tag{45}$$

where

$$B_1 = \pi\int_{\pi/2}^{\pi} B(\theta)\cos^2\theta\sin^2\theta\, d\theta$$
$$\times\, w'|\tilde{w}'|^2 + \tilde{w}'|w'|^2 - w|\tilde{w}|^2 - \tilde{w}|w|^2$$
$$= \left[w + \alpha(\alpha\cdot U)\right]\left[|\tilde{w}|^2 + (\alpha\cdot U)^2 - 2(\alpha\cdot\tilde{w})(\alpha\cdot U)\right]$$
$$+ \left[\tilde{w} - \alpha(\alpha\cdot U)\right]\left[|w|^2 + (\alpha\cdot U)^2 + 2(\alpha\cdot w)(\alpha\cdot U)\right]$$
$$- w|\tilde{w}|^2 - \tilde{w}|w|^2$$
$$= (w + \tilde{w})(\alpha\cdot U)^2 + 2(\alpha\cdot U)\left[(\alpha\cdot w)\tilde{w} - (\alpha\cdot\tilde{w})w\right]$$
$$+ (\alpha\cdot U)(|\tilde{w}|^2 - |w|^2)\alpha - 2(\alpha\cdot U)^2\left[\alpha\cdot(\tilde{w} + w)\right]\alpha$$

Using the formulas (43), (44) and the lemma, we have

$$\int\left[w'|\tilde{w}'|^2 + \tilde{w}'|w'|^2 - w|\tilde{w}|^2 - \tilde{w}|w|^2\right]d\epsilon$$
$$= 2\pi\cos^2\theta|U|^2(w + \tilde{w}) + 4\pi\cos^2\theta\left[(U\cdot w)\tilde{w} - (U\cdot\tilde{w})w\right]$$
$$+ 2\pi\cos^2\theta(|\tilde{w}|^2 - |w|^2)U - 2\pi\cos^2\theta|U|^2$$
$$\left\{\sin^2\theta(w + \tilde{w}) + (2 - 3\sin^2\theta)(1/|U|^2)\left[(\tilde{w} + w)\cdot U\right]U\right\}$$
$$= -2\pi\cos^2\theta\sin^2\theta|U|^2(\tilde{w} + w) + 6\pi\cos^2\theta\sin^2\theta(|\tilde{w}|^2 - |w|^2)U$$
$$= 2\pi\cos^2\theta\sin^2\theta\left[(2|\tilde{w}|^2 - 4|w|^2 + 2w\cdot\tilde{w})\tilde{w}\right.$$
$$\left. + (2|w|^2 - 4|\tilde{w}|^2 + 2w\cdot\tilde{w})w\right]$$

Therefore,

$$\int f^{(0)}\tilde{f}^{(0)}\left[w'|\tilde{w}'|^2 + \tilde{w}'|w'|^2 - w|\tilde{w}|^2 - \tilde{w}|w|^2\right]B(\theta)\,d\theta\,d\epsilon\,d\tilde{v}$$
$$= 4B_1 n f^{(0)}\left(|w|^2 w - \frac{5KT}{m}w\right) \tag{46}$$

$$|w'|^2|\tilde{w}'|^2 - |w|^2|\tilde{w}|^2$$
$$= |w + \alpha(\alpha\cdot U)|^2|\tilde{w} - \alpha(\alpha\cdot U)|^2 - |w|^2|\tilde{w}|^2$$
$$= (|w|^2 + |\tilde{w}|^2)(\alpha\cdot U)^2 + 2(\alpha\cdot U)\left[|\tilde{w}|^2(\alpha\cdot w) - |w|^2(\alpha\cdot\tilde{w})\right]$$
$$- (\alpha\cdot U)^4 - 4(\alpha\cdot U)^2(\alpha\cdot w)(\alpha\cdot\tilde{w})$$

Using the formulas (43), (44) and the lemma, we have

$$\int\left[|w'|^2|\tilde{w}'|^2 - |w|^2|\tilde{w}|^2\right]d\epsilon$$
$$= 2\pi\cos^2\theta\,|U|^2(|w|^2 + |\tilde{w}|^2) + 4\pi\cos^2\theta\left[|\tilde{w}|^2(U\cdot w) - |w|^2(U\cdot\tilde{w})\right]$$
$$-2\pi\cos^2\theta(1-\sin^2\theta)|U|^4$$
$$-4\pi\cos^2\theta\sin^2\theta\,|U|^2(w\cdot\tilde{w}) - 4\pi\cos^2\theta(2 - 3\sin^2\theta)(U\cdot w)(U\cdot\tilde{w})$$
$$= 2\pi\cos^2\theta\sin^2\theta\left[|U|^4 - 2|U|^2(w\cdot\tilde{w}) + 6(U\cdot w)(U\cdot\tilde{w})\right]$$
$$= 2\pi\cos^2\theta\sin^2\theta\left[|w|^4 + |\tilde{w}|^4 - 4|w|^2|\tilde{w}|^2 + 2(w\cdot\tilde{w})^2\right]$$

Therefore,

$$\int f^{(0)}\tilde{f}^{(0)}\left[|w'|^2|\tilde{w}'|^2 - |w|^2|\tilde{w}|^2\right]B(\theta)\,d\theta\,d\epsilon\,d\tilde{v}$$
$$= 2B_1 n f^{(0)}\left[|w|^4 - 10\frac{KT}{m}|w|^2 + 15\left(\frac{KT}{m}\right)^2\right] \tag{47}$$

Combining the equalities (45), (46), and (47) with the equality (42), we get

$$J\left[g^{(1)}(z',\tilde{z}') - g^{(1)}(z,\tilde{z})\right]$$
$$= B_1 n f^{(0)}\left\{\sum_{1\leqslant i,j\leqslant 3} a_{ij}\left(3w_i w_j - |w|^2\delta_{ij}\right) + 4\sum_{1\leqslant i\leqslant 3} a_{i4}\left(|w|^2 w_i - \frac{5KT}{m}w_i\right)\right.$$
$$\left. + 2a_{44}\left[|w|^4 - 10|w|^2\frac{KT}{m} + 15\left(\frac{KT}{m}\right)^2\right]\right\} \tag{48}$$

Substituting the right-hand sides of (41) and (48) for the corresponding terms on the right-hand side of (29), we have

$$\mathcal{J}[f^{(0)}f^{(1)}] = f^{(0)}\left\{ \sum_{1\leq i,j\leq 3}\left(\frac{m}{KT}\frac{\partial u_j}{\partial x_i} - 3B_1 n a_{ij}\right)\left(w_i w_j - \frac{1}{3}|w|^2\delta_{ij}\right) + \sum_{1\leq i\leq 3}\left(\frac{m}{2KT^2}\frac{2T}{\partial x_i} - 4B_1 n a_{i4}\right)\left(|w|^2 w_i - \frac{5KT}{m}w_i\right) - 2B_1 n a_{44}\left[|w|^4 - 10|w|^2\frac{KT}{m} + 15\left(\frac{KT}{m}\right)^2\right]\right\} \tag{49}$$

C. S. Wang-Chang and G. E. Uhlenbeck have studied the eigenvalue problem for the linear Boltzmann integral operator. In particular, the three terms on the right-hand side of (49) are eigenfunctions with the eigenvalues $-6B_1n$, $-4B_1n$, and $-4B_1n$ respectively. Therefore, a special solution of the equation (49) is

$$f^{(1)} = f^{(0)}\left\{ \sum_{1\leq i,j\leq 3}\left(\frac{a_{ij}}{2} - \frac{m}{6B_1nKT}\frac{\partial u_j}{\partial x_i}\right)\left(w_i w_j - \frac{1}{3}|w|^2\delta_{ij}\right) + \sum_{1\leq i\leq 3}\left(a_{i4} - \frac{m}{8B_1nKT^2}\frac{\partial T}{\partial x_i}\right)\left(|w|^2 w_i - \frac{5KT}{m}w_i\right) + \frac{a_{44}}{2}\left[|w|^4 - 10|w|^2\frac{KT}{m} + 15\left(\frac{KT}{m}\right)^2\right]\right\} \tag{50}$$

Obviously, this solution satisfies the additional integral conditions.

7. THE FIRST-ORDER APPROXIMATE MACROSCOPIC EQUATIONS FOR TURBULENT FLOWS

Setting

$$q_i^{(n)} = \frac{m}{2}\int |w|^2 w_i f^{(n)}\, dv$$

$$P_{ij}^{(n)} = m\int w_i w_j f^{(n)}\, dv$$

we have

$$q_i^{(0)} = 0$$

$$q_i^{(1)} = -\lambda\left(\frac{\partial T}{\partial x_i} - \frac{8KT^2}{m}B_1 n a_{i4}\right)$$

$$P_{ij}^{(0)} = nKT\delta_{ij}$$

$$P_{ij}^{(1)} = \mu\left[\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{6B_1nKT}{m}\frac{a_{ij}+a_{ji}}{2}\right) - \frac{2\delta_{ij}}{3}\left(\frac{\partial u_k}{\partial x_k} - \frac{3B_1nKT}{m}a_{kk}\right)\right]$$

where

$$\lambda = \frac{5K^2T}{8B_1m}, \qquad \mu = \frac{KT}{6B_1}$$

are the thermal conductivity and the viscosity, respectively, and the position variables x and $\hat{x}$ in $a_{i4}, a_{ij}, \ldots$ are equal: $x = \hat{x}$. On account of (37) and (35), we have

$$q_i^{(0)} + q_i^{(1)} = -\lambda\left[\frac{\partial T}{\partial x_i} - \frac{4B_1m}{3nK^2T}\left(R^{(1,2)}_{i} - \frac{3KT}{m}R^{(1,0)}_{i}\right)\right] \tag{51}$$

$$P_{ij}^{(0)} + P_{ij}^{(1)} = -\mu\left\{\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{3B_1m}{nKT}\left(R^{(1,1)}_{i,j} + R^{(1,1)}_{j,i}\right)\right] - \frac{2\delta_{ij}}{3}\left(\frac{3nKT}{2\mu} + \frac{\partial u_k}{\partial x_k} - \frac{3B_1m}{nKT}R^{(1,1)}_{k,k}\right)\right\} \tag{52}$$

Substituting the right-hand sides of the equalities (51) and (52) for q_i and P_{ij} in equations (18) and (19), we get the following approximate macroscopic equations:

$$\frac{Dn}{Dt} + n\frac{\partial u_i}{\partial x_i} = 0 \tag{53}$$

$$mn\frac{Du_i}{Dt} - \frac{\partial}{\partial x_j}\left\{\mu\left[\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{3B_1m}{nKT}\left(R^{(1,1)}_{i,j} + R^{(1,1)}_{j,i}\right)\right) - \frac{2\delta_{ij}}{3}\left(\frac{3nKT}{2\mu} + \frac{\partial u_k}{\partial x_k} - \frac{3B_1m}{nKT}R^{(1,1)}_{k,k}\right)\right]\right\} = 0$$

$$(1 \leqslant i \leqslant 3) \tag{54}$$

$$\frac{DT}{Dt} - \frac{2}{3nK}\left\{\mu\left[\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{3B_1m}{nKT}\left(R^{(1,1)}_{i,j} + R^{(1,1)}_{j,i}\right)\right) - \frac{2\delta_{ij}}{3}\left(\frac{3nKT}{2\mu} + \frac{\partial u_k}{\partial x_k} - \frac{3B_1m}{nKT}R^{(1,1)}_{k,k}\right)\right]\frac{\partial u_i}{\partial x_j} + \frac{\partial}{\partial x_i}\left[\lambda\left(\frac{\partial T}{\partial x_i} - \frac{4B_1m^2}{3nK^2T}\left(R^{(1,2)}_{i} - \frac{3KT}{m}R^{(1,0)}_{i}\right)\right)\right]\right\} = 0 \tag{55}$$

Sometimes nKT in the above equations will be denoted by p. By virtue of $x = \hat{x}$, we have $R^{(1,1)}_{i,j} = R^{(1,1)}_{j,i}$. The expression $R^{(1,1)}_{i,j} + R^{(1,1)}_{j,i}$ is, therefore, equal to $2R^{(1,1)}_{i,j}$.

Since

$$g^{(1)} = f^{(0)}\hat{f}^{(0)} \sum_{0 \leqslant i, j \leqslant 4} a_{ij}\psi_i\hat{\psi}_j$$

we have

$$R_{ik}^{(2,0)(0)} = \tfrac{1}{3} R^{(2,0)} \delta_{ik}$$

$$R_{ik,j}^{(2,1)(0)} = \tfrac{1}{3} R^{(2,1)}_{j} \delta_{ik}$$

$$R_{ik}^{(2,2)(0)} = \tfrac{1}{3} R^{(2,2)} \delta_{ik}$$

$$R^{(3,0)(0)}_{k} = \frac{5KT}{m} R^{(1,0)}_{k}$$

$$R^{(3,1)(0)}_{k,j} = \frac{5KT}{m} R^{(1,1)}_{k,j}$$

$$R^{(3,2)(0)}_{k} = \frac{5KT}{m} R^{(1,2)}_{k}$$

Substituting the right-hand sides of the above equalities for the corresponding R's in the conservation equations (20)–(25), we obtain the following 25 equations governing the evolution of the 25 R's:

$$\frac{\mathcal{D} R^{(0,0)}}{\mathcal{D} t} + \left(\frac{\partial u_k}{\partial x_k} + \frac{\partial \hat{u}_k}{\partial \hat{x}_k} \right) R^{(0,0)} + \frac{\partial R^{(1,0)}_{k}}{\partial x_k} + \frac{\partial R^{(0,1)}_{k}}{\partial \hat{x}_k} = 0 \tag{56}$$

$$\frac{\mathcal{D} R^{(1,0)}_{i}}{\mathcal{D} t} + \left(\frac{\partial u_k}{\partial x_k} + \frac{\partial \hat{u}_k}{\partial \hat{x}_k} \right) R^{(1,0)}_{i} + \frac{D u_i}{D t} R^{(0,0)} + R^{(1,0)}_{k} \frac{\partial u_i}{\partial x_k}$$

$$+ \frac{1}{3} \frac{\partial R^{(2,0)}}{\partial x_i} + \frac{\partial R^{(1,1)}_{i,k}}{\partial \hat{x}_k} = 0 \qquad (1 \leqslant i \leqslant 3) \tag{57}$$

$$\frac{\mathcal{D} R^{(1,1)}_{i,j}}{\mathcal{D} t} + \left(\frac{\partial u_k}{\partial x_k} + \frac{\partial \hat{u}_k}{\partial \hat{x}_k} \right) R^{(1,1)}_{i,j} + \frac{D u_i}{D t} R^{(0,1)}_{j} + \frac{\hat{D} \hat{u}_j}{\hat{D} t} R^{(1,0)}_{i} + R^{(1,1)}_{k,j} \frac{\partial u_i}{\partial x_k}$$

$$+ R^{(1,1)}_{i,k} \frac{\partial \hat{u}_j}{\partial \hat{x}_k} + \frac{1}{3} \frac{\partial R^{(2,1)}_{j}}{\partial x_i} + \frac{1}{3} \frac{\partial R^{(1,2)}_{i}}{\partial \hat{x}_j} = 0 \qquad (1 \leqslant i, j \leqslant 3) \tag{58}$$

$$\frac{\mathcal{D} R^{(2,0)}}{\mathcal{D} t} + \left(\frac{5}{3} \frac{\partial u_k}{\partial x_k} + \frac{\partial \hat{u}_k}{\partial \hat{x}_k} \right) R^{(2,0)} + 2 \frac{D u_i}{D t} R^{(1,0)}_{i} + \frac{5K}{m} \frac{\partial (T R^{(1,0)}_{k})}{\partial x_k}$$

$$+ \frac{\partial R^{(2,1)}_{k}}{\partial \hat{x}_k} = 0 \tag{59}$$

$$\frac{\mathfrak{D} R^{(2,1)}_{j}}{\mathfrak{D} t} + \left(\frac{5}{3}\frac{\partial u_k}{\partial x_k} + \frac{\partial \hat{u}_k}{\partial \hat{x}_k}\right) R^{(2,1)}_{j} + 2\frac{Du_i}{Dt} R^{(1,1)}_{i,j} + \frac{\hat{D}\hat{u}_j}{\hat{D}t} R^{(2,0)} + R^{(2,1)}_{k} \frac{\partial \hat{u}_j}{\partial \hat{x}_k}$$

$$+ \frac{5K}{m}\frac{\partial\left(TR^{(1,1)}_{k,j}\right)}{\partial x_k} + \frac{1}{3}\frac{\partial R^{(2,2)}}{\partial \hat{x}_j} = 0 \qquad (1 \leqslant j \leqslant 3) \tag{60}$$

$$\frac{\mathfrak{D} R^{(2,2)}}{\mathfrak{D} t} + \frac{5}{3}\left(\frac{\partial u_k}{\partial x_k} + \frac{\partial \hat{u}_k}{\partial \hat{x}_k}\right) R^{(2,2)} + 2\frac{Du_i}{Dt} R^{(1,2)}_{i} + 2\frac{\hat{D}\hat{u}_j}{\hat{D}t} R^{(2,1)}_{j}$$

$$+ \frac{5K}{m}\left[\frac{\partial\left(TR^{(1,2)}_{k}\right)}{\partial x_k} + \frac{\partial\left(\hat{T}R^{(2,1)}_{k}\right)}{\partial \hat{x}_k}\right] = 0 \tag{61}$$

To save space, we have omitted the equations which are symmetric in form to the equations (57), (59), and (60), respectively. There are altogether 30 equations in (53)–(61), which closely govern the 30 unknowns: n, u, T and 25 R's. On account of the symmetry, the number of equations can be reduced to 20. They are the simplest equations governing the motion of the turbulent flow. Therefore, they are the equations in hydrodynamics worthy of being explored next to the Navier–Stokes equations.

If the flow is incompressible and μ is constant, the equations of motion can be reduced to the following form:

$$\frac{\partial u_i}{\partial x_i} = 0$$

$$\frac{Du_i}{Dt} - \Delta \nu u_i + \frac{\partial p}{\partial x_i} + \frac{1}{n^2}\frac{\partial\left(R^{(1,1)}_{i,j}\right)}{\partial x_j} - \frac{1}{3n^2}\frac{\partial}{\partial x_i}\left(R^{(1,1)}_{k,k}\right) = 0 \qquad \left(\nu = \frac{\mu}{\rho}\right)$$

$$\frac{\partial R^{(1,1)}_{i,k}}{\partial \hat{x}_k} = 0 \qquad (1 \leqslant i \leqslant 3)$$

$$\frac{\mathfrak{D} R^{(1,1)}_{i,j}}{\mathfrak{D} t} + R^{(1,1)}_{k,j}\frac{\partial u_i}{\partial x_k} + R^{(1,1)}_{i,k}\frac{\partial \hat{u}_j}{\partial \hat{x}_k} + \frac{1}{3}\left(\frac{\partial R^{(2,1)}_{j}}{\partial x_i} + \frac{\partial R^{(1,2)}_{i}}{\partial \hat{x}_j}\right) = 0$$

$$(1 \leqslant i, j \leqslant 3)$$

$$\frac{\partial R^{(2,1)}_{k}}{\partial \hat{x}_k} = 0$$

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